Number Theory by Final Digits

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1 Introduction

I initially began writing this as an easy introduction into competition number theory, specifically that on problems that ask a contestant to find the last digit of some (often gross) expression. Doing so requires some basic properties of modular arithmetic, which is not too difficult. However, when the question asks for increasingly more digits, we will need increasingly more advanced techniques. When pondering this, I realized that a summary on finding (arbitrarily many) final digits of expressions actually uses almost all[1] of number theory relevant to lower level competitions.

This is good for multiple reasons; I always find that learning math through the lens of concrete problems makes the math more easier to understand and remember. More directly, these problems are the most commonly seen number theory problems at district meets.

I am still maintaining the goal of being a relatively easy introduction, but the scope is much greater. While I will make very few assumptions on prior knowledge to keep this self-contained, I may not go into depth on topics that are not strictly relevant. I hope you enjoy reading this as much as I did writing it!

2 Simplification and The Language of Remainders

Here's a simple example.

(My butt in conjunction with RANDOM.ORG)

Find the last digit of

$$17826 + 74347 + 27478 + 91320.$$

Okay, calculate the expression and then I'll tell you the last digit. I'm waiting.

Maybe summing these numbers wasn't so bad, but what if I gave each number an additional digit? Or two? Or fifteen? Clearly this method doesn't scale up very well, and computing these sums take time anyways. What we want is a faster method, one that can potentially scale, as well. This is a motivation for modular arithmetic – the language of remainders.

The idea or technique you may recognize is that since the question only asks for the *last digit* of the sum, we can just take the last digits of all the numbers we're adding up, take that sum of those digits, and then take the last digit of that sum. This is "legal" in a sense since the all the other digits that aren't the last don't change the value of the last digit. For instance, $24 + 123$ has the last digit 7 as does $1231234 + 3$, even though the two sums have completely different numbers; their last digits are the same, so the sum should have the same last

[1] We miss factoring for the most part and much of Diophantine equations.

digit as well. In a sense, when asking for the last digit of an expression, we should only work with the last digits of the numbers involved.

Note that taking the last digit of a number is the same thing as taking the remainder of the number when divided by 10. And to express a numbers remainder when divided by another, we introduce an operator known as the **modulus**, hence the name *modular* arithmetic. The syntax of the statement is as follows; given the verbal statement "a number a, when divided by a number n, has remainder b:"

$$a \mod (n) = b.$$

This has the advantage of being significantly quicker than writing the verbal statement.

Another common statement is to say that two numbers have the same remainder when divided by a certain number. In this case, it is written as

$$a \equiv b \mod n.$$

For instance,

$$9 \equiv 43 \mod 17.$$

Let's now translate our problem into the language of remainders. We are asked to compute

$$17826 + 74347 + 27478 + 91320 \mod 10.$$

Again, since we only care about the last digit of the numbers we are adding up, we can just add their last digits. And again, the last digit of a number is the same as its remainder when divided by 10, so

$$\begin{aligned} 17826 + 74347 + 27478 + 91320 \mod 10 \equiv & ((17826 \mod 10) \\ & + (74347 \mod 10) \\ & + (27478 \mod 10) \\ & + (91320 \mod 10)) \mod 10 \\ \equiv & \, 6 + 7 + 8 + 0 \mod 10 \\ \equiv & \, 21 \mod 10 \\ = & \, 1. \end{aligned}$$

This demonstrates that if we want to the find the remainder of a sum, all we have to do is add the remainders together, then take the remainder one final time at the end. We can say the modulus operator is compatible with addition.

In an actual competition setting, the first line is usually implicit and not written.

Let's look at some more examples:

(My butt in conjunction with RANDOM.ORG)

Find the last digit of

$$4354 \times 1174.$$

We can rewrite this problem as asking us to compute the product mod 10 or

$$4354 \times 1174 \mod 10.$$

Again, since we only care about the last digit of the product, we can look to take the remainders of our factors, and then multiply those two, then take the remainder again (just in case).

$$\begin{aligned} 4354 \times 1174 \mod 10 &\equiv ((4354 \mod 10) \times (1174 \mod 10)) \mod 10 \\ &\equiv (4 \times 4) \mod 10 \\ &\equiv 16 \mod 10 \\ &= 6. \end{aligned}$$

This goes to show that the modulus operator is compatible with multiplication as well.

We have to be a bit careful, however, with division and exponentiation. We will address those in later sections.

2.1 Examples

1. Find the sum of 31 and 148 in modulo 24. (Brilliant Example 3.3)
2. Find the remainder when $123 + 234 + 32 + 56 + 22 + 12 + 78$ is divided by 3. (Brilliant Example 3.4)
3. What is $(8 \times 16) \mod 7$. (Brilliant Example 4.2)
4. Find the remainder when $124 \cdot 134 \cdot 23 \cdot 49 \cdot 235 \cdot 13$ is divided by 3. (Brilliant Example 4.3)

3 Painting Patterns

This a kind of problem you see somewhat frequently,

> (Brilliant Example 1.1)
>
> Find the last digit of
> $$7^{358}.$$

We can continue our simplifying method of only looking at the last digit – the number's remainder when divided by 10. The problem then transforms to computing

$$7^{358} \mod 10.$$

Unfortunately, $7 \mod 10 = 7$, so we can't really go any further with this. And the modulus operator doesn't let us just take $358 \mod 10$ and replace it. Remember, exponentiation and division are a bit different; it is only addition and multiplication that lets us replace things easily.

We will have to defer to the fact that exponentiation is repeated multiplication. To do so, we continuously multiply by 7, take its last digit, then multiply that by 7 and repeat.

In order to a get a “higher” power of 7, we can multiply a smaller one by 7. Since we only care about the final digit, we can take the smaller one’s last digit (or remainder when divided by 10) and multiply that by 7 instead. Then in case that product is greater than 10, we take that numbers last digit (or remainder when divided by 10) to get its last digit.

$$\begin{aligned}
7^1 &\mod 10 \equiv 7 \mod 10 &&= 7\\
7^2 &\mod 10 \equiv (7 \times 7^1 \mod 10) \mod 10 \equiv (7 \times 7) \mod 10 &&= 9\\
7^3 &\mod 10 \equiv (7 \times 7^2 \mod 10) \mod 10 \equiv (7 \times 9) \mod 10 &&= 3\\
7^4 &\mod 10 \equiv (7 \times 7^3 \mod 10) \mod 10 \equiv (7 \times 3) \mod 10 &&= 1\\
7^5 &\mod 10 \equiv (7 \times 7^4 \mod 10) \mod 10 \equiv (7 \times 1) \mod 10 &&= 7
\end{aligned}$$

A lot of the calculation above is unnecessary, and exists only to be thorough. For instance, the third line can be simplified as

$$7^3 \mod 10 \equiv 7 \times 3 \mod 10 = 1 \mod 10,$$

or even

$$7^3 \equiv 1 \mod 10$$

Wait a minute! We already calculated $7 \times 7 \mod 10$, and we know its 9. And then we see we also already calculated $9 \times 7 \mod 10$ to be 3. A pattern emerges; it starts over once the final digit hits 1. It seems that the final digits of powers of 7 cycle through a set of four numbers, or

$$\text{Powers of } 7 \mod 10 \in \{7, 9, 3, 1\}.$$

So we know that the last digit of 7^{358} will be either 7, 9, 3 or 1. We’ve narrowed it down a little bit, but we can do better. We know that it’ll be 7, 9, 3 or 1, but which one? We will utilize the fact that the final digits aren’t just randomly chosen from the set, no, the final digits **cycle.**

Let’s pair the exponent with the final digit. The final digit of 7^1 is 7, 7^2 is 9, 7^3 is 3 and 7^4 is 1. Putting this in table form yields the following.

Exponent of 7^k	$7^k \mod 10$
1	7
2	9
3	3
4	1
5	7
6	9
7	3
8	1
$\vdots$	$\vdots$

Note that it goes in a period of 4; every 4 exponents, the pattern starts over again. Taking a closer look, if the exponent is a multiple of 4, the final digit of 7^k is 1. Using this logic, if the exponent is 1 greater than a multiple of 4, the final digit of 7^k is 7; if the exponent is 2 greater than a multiple of 4, the final digit of 7^k is 9 and so on. So we can generalize this table:

Exponent of 7^k	$7^k \mod 10$
k is a multiple of $4 + 0$	1
k is a multiple of $4 + 1$	7
k is a multiple of $4 + 2$	9
k is a multiple of $4 + 3$	3

We can then further simplify the "k is a multiple of 4 + something" through modular arithmetic. Note that if a number is a multiple of 4, its remainder when divided by 4 should be 0. Likewise, if a number is greater than a multiple of 4 by 1, its remainder when divided by 4 should be 1 and so on. Using this logic, we can further simplify the table.

Exponent of 7^k	$7^k \mod 10$
$k \equiv 0 \mod 4$	1
$k \equiv 1 \mod 4$	7
$k \equiv 2 \mod 4$	9
$k \equiv 3 \mod 4$	3

All that's left is to find what the exponent of 7^{358} is mod 4. By division, $358 \equiv 2 \mod 10$, which says $7^{358} \mod 10 = 9$ making its final digit 9. And we are done.

Again, in a competition setting most of the calculations and formatting done here is unnecessary and impractical. For a question like this, I would write a list of the last digits of powers of 7 starting from 1, and writing the exponent below. From there I would mentally note the pattern, then find the remainder of the expression modulo 4. My work might look something like this. Things in parenthesis are things I do not write down, but I have included here so the work makes sense.

(Exponent of 7^k)	($7^k \mod 10$)
1	7
2	9
3	3
4	1

$$358 \equiv 2 \mod 10 \to 9.$$

I don't go further because we can simply assume that things tend to cycle when it hits 1.

I initially tried to Google 7^{358} and then show its last digit to show that we are correct, but this is a number with $\left\lfloor \log_{10}\left(7^{358}\right)\right\rfloor = 302$ digits, so you'll just have to trust me on this one.

3.1 Examples

1. Find the last digit of 2^{2016}. (Brilliant Example 1.2)
2. Consider a number 3^n where n is a positive integer.

   If $n = 2016$, the last digit of 3^n is a.

   If $n = 9018$, the last digit of 3^n is b.

   What is $a + b$? (Brilliant Try It Yourself 1.2)
3. Find the last digit of 17^{17}. (Brilliant Example 1.3)

4 Systems and Gazing into the Abyss

Finding the last digit of most expressions is fairly straightforward due to the fact that computing things mod 10 isn't too bad; we can just do a couple of brute force calculations and then find the pattern. However, this is not so easy when we no longer look for only the last digit. When we look for the last n digits, we have to use a new chest of tricks.

(Brilliant Example 3.1)

Find the last two digits of

$$74^{540}.$$

This is equivalent to finding

$$74^{540} \mod 100.$$

Actually finding it, however, might take a bit more effort.

Our process of repeatedly taking $74^k \mod 100$ here isn't going to work because constantly multiplying a number by 74 is awful, even if we only care about the last two digits.

So clearly taking the remainder when dividing by 100 isn't an option, so maybe we should look for remainders when dividing by more reasonable numbers. We can't just pick numbers haphazardly we should pick our divisors to be numbers that are "part" of 100, so we can combine our remainders in the end someway. Since the modulus is intimately connected with multiplication and division, we should pick numbers that multiply together to yield 100. As

$$100 = 2^2 \cdot 5^2,$$

it would make perfect sense to use $2^2 = 4$ and $5^2 = 25$. Indeed, calculating the remainder of a number when divided by 4 and 25 is much less painful than the remainder when divided by 100.

So we look to find

$$74^{540} \mod 4$$

and

$$74^{540} \mod 25.$$

Finding $74^{540} \mod 4$ isn't too bad. Noting that $74 \mod 4 = 2$, this reduces to finding $2^{540} \mod 4$.

We note that we are now interested in looking at the remainder of powers of 2 when divided by 4. This uses the exact same process as that of finding final digits except we can't just extract the final digit; we have to take the remainder when divided by 4.

$$\begin{aligned} 2^1 &\mod 4 = 2 \\ 2^2 &\mod 4 = 0 \\ 2^3 &\mod 4 = 0. \end{aligned}$$

Well, multiplying another number by 0 is, well, zero by the zero product property of multiplication, so we see that any power of 2 with an exponent greater than 1 has remainder of 0 modulo 4. [2]

So now that we know $74^{540} \mod 4 = 0$, we now have to compute $74^{540} \mod 25$.

By noting that $74 \mod 25 = 24 \mod 25$, this simplifies (though not by much...) to finding $24^{540} \mod 25$.

We can make this simpler still by taking advantage of the fact that adding (or subtracting!) a multiple to a number does not change the remainder. That is, if

$$43 \equiv 20 \mod 23,$$

adding 20 shouldn't change the remainder, or

$$43 + 20 \mod 23 \equiv 20.$$

[2] We can also show this by noting that any power of two starting from 2 is a multiple of 4, so it has to have remainder 0 when divided by 4.

So we subtract 25 to get

$$24^{540} \mod 25 \equiv (-1)^{540} \mod 25.$$

We can justify this by saying that a number being 24 greater than a multiple of 25 is the same thing as saying its 1 *less* than a multiple of 25.

Repeating the process of repeatedly taking the exponent of our base modulo a number, we see that

$$\begin{aligned}(-1)^1 &= \mod 25 = -1\\ (-1)^2 &= \mod 25 = 1.\end{aligned}$$

So the powers of (-1) modulo 25 go in the cycle $\{-1, 1\}$. By noting 540 mod $2 = 0$, we get that $74^{540} \mod 25 = 1$.

Putting these two statements give us a **system of linear congruences**. If the last two digits of 74^{540} are some number x, we can express this by saying

$$\begin{aligned}x &\equiv 0 \mod 4\\ &\equiv 1 \mod 25.\end{aligned}$$

Sometimes this can be written as

$$x \equiv \begin{cases}0 & \mod 4\\ 1 & \mod 25.\end{cases}$$

This should look familiar to a system of (linear) equations to you. We will use an essentially modified version of substitution to solve this system.

The first question we should ask is if a solution to the system exists. Just like how in a system of (linear) equations, solutions may not exist, there may not always be a solution to a system of linear congruences. While the checking process is usually somewhat involved, we can just use the existence criterion of **The Chinese Remainder Theorem**: if the system of linear congruences have co-prime divisors, a solution has to exist.

Since 4 and 25 are co-prime, a solution has to exist.[3]

We can do a list search by means of listing all solutions to $x \equiv 1 \mod 25$ (26, 51, 76, 1) and looking for one that has remainder 0 when divided by 4, but there exists an analytic approach, as well.

We proceed by iterative substitution, statement by statement, starting with the first one, namely $x \equiv 0 \mod 4$. Verbally, we can translate this to mean that the number x has remainder 0 when divided by 4. Since multiples of 4 have remainder 0 when divided by 4, we can then conclude x is a multiple of 4, which means that its 4 times some integer or

$$x = 4k; \quad k \in \mathbb{Z}.$$

[3] A more hand-wavy justification is that literally any number has to have final digits, so a solution has to exist.

We can then substitute this $x = 4k$ identity into our system of linear congruences, namely $x \equiv 1 \mod 25$ which precipitates

$$4k \equiv 1 \mod 25; \quad k \in \mathbb{Z}.$$

We want to solve for k; our first thought should be to multiply both sides of the equation by $\frac{1}{4}$, but the statement $\frac{1}{4} \mod 25$ doesn't even make sense. For relatively small numbers (4 and 25), we can keep adding 25 to both sides (which doesn't change the value) until the right hand side is divisible by 4 and then dividing both sides by 4. [4] This is terrible for larger numbers because we might have to add the modulus many, many times and so we will again have to learn new techniques to make this process less arduous.

4.1 Diamonds in the Rough and Modular Multiplicative Inverses

Currently we have that

$$4k \equiv 1 \mod 25.$$

We want to find a way to "divide by 4" such that we can get that k is equal to some other integer modulo 25, or

$$k \equiv m \mod 25; \quad k, m \in \mathbb{Z}.$$

By some advanced mathematics, we note that if the number and the divisor are co-prime, we can "invert" this multiplication. In this example, since 4 and 25 are co-prime, we are able to invert the multiplication by 4 to isolate k. The way we do it is to multiply both sides by a special number that inverts multiplication aptly named the **modular multiplicative inverse**.

That is, if we have the expression

$$a \equiv b \mod n,$$

we define the modular multiplicative inverse a^{-1} such that

$$\left(a \cdot a^{-1}\right) \mod n = 1,$$

which exists if and only if a and n share no common factors, or

$$\gcd(a, n) = 1.$$

We will find the modular multiplicative inverse by what is known as the **extended Euclidean algorithm**, which of course begins with the regular **Euclidean Algorithm**.

We start with our two numbers, in this case 4 and 25. We will the larger number by the smaller one to get a quotient and a remainder. We then divide

[4] In this case, you add 25 thrice to get $4k + 75 \equiv 1 + 75 \mod 25$ which becomes $4k \equiv 76 \mod 25$ which then, once divided, becomes $k \equiv 19$.

the divisor with the remainder, then repeat until the remainder is 0. It is best explained with an example. First we divide 25 by 4. This yields

$$25 = 6(4) + 1.$$

We then divide the divisor (4) by the remainder (1).

$$4 = 4(1) + 0.$$

Since the final remainder is 0, our algorithm is complete. We note the last number inside the parenthesis as the greatest common divisor, which verifies our observation that $\gcd(25, 4) = 1$.

In cases where the algorithm takes multiple steps, we iterate and keep replacing the number to be divided by the number inside the parenthesis and divide it by the remainder until we the remainder is 0, at which point the final number inside the parenthesis is the greatest common divisor.

We now use **Bezout's Identity** which says that for any two integers, we can express their greatest common divisor as a **linear combination** of the two integers. That is, given two numbers a and b,

$$ax + by = \gcd(a, b); \quad a, b, x, y \in \mathbb{Z}.$$

In our case, our equation is given by

$$4x + 25y = 1; \quad x, y \in \mathbb{Z}.$$

If $\gcd(a, b) = 1$, then x is the modular multiplicative inverse of a modulo b, which is what we're looking for. To actually find x and this linear combination, we look to use the steps we created in finding the gcd.

We take the "step" that has remainder 1, in this case $25 = 6(4) + 1$ and solve for the remainder. Here, the process is straightforward and we get

$$1 = 25 - 6(4).$$

or

$$-6(4) + 1(25) = 1.$$

Which tells us that the modular multiplicative inverse of 4 modulo 25 is -6.

In instances where it takes more steps, one has to continuously back substitute more expressions for the remainder until we "have" both relevant numbers. In those cases, too, the simplification process will also take longer.

Again, recall that the modular multiplicative inverse of an integer modulo another one is where the product of the two evaluates to 1. As such, we multiply both parts of the linear congruence $4k = 1 \mod 25$ by -6, which turns the left side to 1 times k. Doing so gives us

$$\begin{aligned}(-6 \cdot 4)k &= 1 \cdot -6 \mod 25\\ k &= -6 \mod 25\\ k + 25 &= -6 + 25 \mod 25\\ k &= 19 \mod 25\end{aligned}$$

We can then rewrite the final statement as saying that k is 19 greater than some multiple of 25, or

$$k = 25l + 19; l \in \mathbb{Z}.$$

Recall pages ago we described the solution x as being four times k or $x = 4k$. Combining these two statements yields

$$\begin{aligned} x &= 4(25l + 19); \quad l \in \mathbb{Z} \\ &= 100l + 76; \quad l \in \mathbb{Z}. \end{aligned}$$

This is equivalent to saying x is 76 greater than an integer multiple of 100. Letting $l = 0$ gives that $x = 0 + 76 = 76$, which finishes the problem.[5]

Also note that finding a multiplicative modular inverse is usually unnecessary for finding the last *two* digits of some expression, as one usually doesn't have to check more than three cases. In cases where it's four digits or greater, it's more practical to do so.

4.2 A Vanishing Act – Euler's Totient Theorem

We got lucky in a way; the patterns of powers of negative 1 are really easy to work with. In certain instances, it may not be as easy.

(Brilliant Example 4.1)

Find the last two digits of

$$33^{42}.$$

It should probably be second nature to note that this problem is logically equivalent to finding

$$33^{42} \mod 100.$$

Following the procedure the section before would suggest to break the calculation into solving a system of linear congruences, but this forces us to find 17^{42} mod 25 which isn't easy to calculate whatsoever. In this case (and actually most cases, from now on), we look to simplify the exponent, too, as much as we can before breaking out a system.

First, we define **Euler's totient function**, $\phi(n)$, as the number of integers less than n that are co-prime to n. Listing numbers is really gross when n is large, so we use what's known as **Euler's product formula**

$$\phi(n) = n \times \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_3}\right),$$

[5] We can also observe that the last *two* digits of a number should not ever exceed 100 or ever be negative.

where $p_1, p_2, \ldots, p_3$ are the unique prime factors of n.[6] Another formulation is given by

$$\phi(n) = n \times \left(\frac{p_1 - 1}{p_1}\right)\left(\frac{p_2 - 1}{p_2}\right) \ldots \left(\frac{p_n - 1}{p_n}\right),$$

which can be shown by algebraically manipulating terms in Euler's product formula.

With that out of the way, we have **Euler's totient theorem** which states that if given two integers a and n co-prime,

$$a^{\phi(n)} \equiv 1 \mod n.$$

Although the $\equiv 1 \mod n$ looks tempting to transform the expression into one that yields a modular multiplicative inverse, doing so is usually slower than using the extended Euclidean algorithm.

Noting that the problem has us work in mod 100, we should then calculate $\phi(100)$, which is given by

$$\begin{aligned}\phi(100) &= 100\left(\frac{1}{2}\right)\left(\frac{4}{5}\right)\\ &= 40.\end{aligned}$$

Then, by Euler's totient theorem,

$$33^{40} \equiv 1 \mod 100.$$

This helps a bit. By exponent rules $33^{42} = 33^{40+2}$, so $33^{42} = 33^{40} \cdot 33^2$. Putting this altogether,

$$\begin{aligned}33^{42} \mod 100 &= 33^{40} \times 33^2 \mod 100\\ &= 1 \times 33^2 \mod 100\\ &\equiv 33^2 \mod 100.\end{aligned}$$

This is much easier as all we have to do now is compute $33^2 \mod 100$ which comes out to be $1089 \mod 100$, or 89. This finishes the problem.

Euler's totient theorem works to reduce the exponent, but sometimes alone is insufficient for much larger exponents, where we have exponents in the exponent itself. In these cases, we may have to use Euler's totient theorem multiple times.

(Brilliant Example 4.2)

Find the last three digits of

$$4^{2^{43}}.$$

This is the same as

$$4^{2^{42}} \mod 1000.$$

[6] This can be proved through the inclusion-exclusion principle and fundamental theorem of arithmetic.

While we may be tempted to immediately apply Euler's totient theorem to attack the exponent, we are unable to as 4 and 1000 are not co-prime. We proceed by introducing a system right away to address this.

We have

$$4^{2^{42}} \equiv 0 \mod 8,$$

and now we have to find

$$4^{2^{42}} \mod 125$$

before solving the system as a whole. And here, since 4 and 125 are co-prime, we can make use of Euler's totient theorem. As we are working modulo 125, we calculate $\phi(125) = 100$, so $4^{100} \equiv 1 \mod 125$.

This doesn't initially seem to be helpful, as 4^{100} is a far cry from $4^{2^{43}}$. Our saving grace is to take the exponent, 2^{43}, modulo 100. Why? Note that if we can somehow rewrite 2^{43} as some multiple of 100 plus some remainder, we can "throw out" the multiple of 100 as it equates to

$$4^{100n} \mod 125 \to (1)^n \mod 125 = 1.$$

So now we are interested in finding

$$2^{42} \mod 100.$$

Formally speaking, we should break out a system to solve this. This goes pretty quickly, as we know that $2^{43} \equiv 0 \mod 4$, and all that's left is to find $2^{43} \mod 25$. Using Euler's totient theorem *again*, we find that

$$2^{42} \mod 25 = (2^{20})^2 \cdot 2^2 \mod 25 = 4 \mod 25.$$

We know the system

$$x \equiv \begin{cases} 4 & \mod 25 \\ 0 & \mod 4 \end{cases}$$

has a solution by the Chinese remainder theorem, and is given by $x = 4 \mod 100$. Now we can start digging our way out of the hole. From here, we know that $2^{43} \equiv 4 \mod 100$, so

$$\begin{aligned} 4^{2^{43}} \mod 125 &\equiv (4^{100})^{\text{something}} \times 4^4 \mod 125 \\ &\equiv 1^{\text{something}} \times 256 \mod 125 \\ &\equiv 256 \mod 125 \\ &\equiv 6 \mod 125. \end{aligned}$$

Combining this and $0 \mod 8$ has a solution, namely $256 \mod 1000$, making the last three digits 256, which solves the problem.

5 Examples

1. What are the last three digits of 123^{456}? (Brilliant Try it Yourself 7.1)
2. Find the last four digits of 1444^{144^4} (Brilliant Try it Yourself 7.2)
3. Find the last two digits of $299999^{29999^{2999^{299^{29^2}}}}$. (Brilliant Try it Yourself 7.4)

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